REPORT DOCUMENTATION PAGE	Form Approved OMB No. 0704-0188

1. REPORT DATE (DD-MM-YYYY)	2. REPORT TYPE	3. DATES COVERED (From - To)
06/15/09	Final Technical Report	03/15/06-11/30/08

4. TITLE AND SUBTITLE Curse-of-Dimensionality-Free Computing, Information-Savvy Controllers and UAV Operations	5a. CONTRACT NUMBER FA9550-06-1-0238
	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S) William McEneaney, PhD	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) The Regents of the Univ. of Calif., U.C. San Diego 9500 Gilman Drive, MC 0934 La Jolla, CA 92093-0934	8. PERFORMING ORGANIZATION REPORT NUMBER 27751A
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) AFOSR/PKA USAF, AFRL AF Office of Scientific Research 875 N. Randolph St., Room 3112 Arlington, VA 22203	10. SPONSOR/MONITOR'S ACRONYM(S)
	11. SPONSOR/MONITOR'S REPORT NUMBER(S) AFRL-OSR-VA-TR-2012-0148

12. DISTRIBUTION/AVAILABILITY STATEMENT

Approved For Public Release

13. SUPPLEMENTARY NOTES

14. ABSTRACT

See attached

15. SUBJECT TERMS

max-plus, nonlinear, control, sensor tasking, idempotent, tropical algebra

16. SECURITY CLASSIFICATION OF: a. REPORT	b. ABSTRACT	c. THIS PAGE	17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
Unclassified	Unclassified	Unclassified	UU		William McEneaney
					19b. TELEPHONE NUMBER (Include area code) 858-822-5835

#14 Abstract:

We investigated the use of idempotent (e.g., max-plus) algebraic methods for solution of nonlinear control problems. The main effort used deterministic infinite time-horizon optimal control problems as the vehicle for development of the approach, i.e, we developed the methods for that class of problems as a demonstration of the general approach. We obtained a curse-of-dimensionality-free max-plus numerical method. Combining this new theory with some convex programming based pruning, we demonstrated solution of a particular class on nonlinear problems over six-dimensional space. Standard solution methods would take computational time on the order of decades to solve such a problem, whereas we were able to obtain a solution in under an hour on a desktop machine for the example problem.

We also investigated sensing UAV tasking algorithms. We demonstrated that the correct criterion for success, expected reduction of troop losses, took the specific form of a piece-wise linear concave function over a probability simplex. We further found that this class of problems could also be solved efficiently with idempotent methods. This was unexpected, as previously it was believed that one needed idempotent linearity of the associated semigroup for application of such techniques. The key was found to lie in the idempotent distributive property.

Final Report:
AFOSR Grant FA9550-06-1-0238

Curse-of-Dimensionality-Free Computing, Information-Savvy Controllers and UAV Operations

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1 Introduction

Multiple breakthroughs were made during the period of the effort. We divide them up according to the two main areas.

2 Curse-of-Dimensionality-Free Methods for HJB PDEs

At the outset of the project, we had the basic algorithm in hand for a curse-of-dimensionality-free approach for a class of first-order HJB PDEs where the Hamiltonian is written (or approximated) as a pointwise maximum over a finite set of quadratic forms. Although the approach is generally applicable to such problems, for the purposes of demonstration we needed to choose a particular form. We chose the class of HJB PDEs corresponding to infinite time-horizon, average cost per unit time, deterministic control problems. More exactly, we consider HJB PDEs of the form

$$\begin{aligned} &0 = H(x, \nabla V) \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \\ &V(0) = 0, \end{aligned}$$

where

$$H(x,p) = \max_{m \in \mathcal{M}} H^m(x,p),$$

with $\mathcal{M} \doteq]1, M[\doteq \{1, 2 \dots M\}$ and

$$H^m(x,p) = \tfrac{1}{2}x' D^m x + \tfrac{1}{2}p'\Sigma^m p + (A^m x)'p + (l_1^m)'x + (l_2^m)'p + \alpha^m \quad \forall m \in \mathcal{M}.$$

Note that the solution is **not** a piece-wise linear-quadratic function, but is fully nonlinear.

Although, we had an algorithm, we lacked a convergence analysis and error bounds. This was necessary for full acceptance of the approach. The resulting full analysis is spread across [16] and [15]. In order to give a sense of the results we note that at each step, the solution is represented by a pointwise maximum of quadratics, i.e., at step N, the approximation is

$$V^N(x) = \bigoplus_{k \in \mathcal{K}_N} U_k^N(x)$$

with each U_k^N represented by the coefficients of the quadratic, $(Q_k^N, \overline{x}_k^N, c_k^N)$. At the next step, one has an approximation of the form

$$V^{N+1}(x) = \bigoplus_{m\in\mathcal{M}} S_\tau^m \left[\bigoplus_{k\in\mathcal{K}_N} U_k^N\right](x)$$
$$= \bigoplus_{m\in\mathcal{M}} \bigoplus_{k\in\mathcal{K}_N} S_\tau^m\left[U_k^N\right](x) \doteq \bigoplus_{k\in\mathcal{K}_{N+1}} U_k^{N+1}(x),$$

where each semigroup, S_τ^m, approximates the action of a single linear-quadratic control problem associated with H^m. The computations are reduced to analytical (modulo a matrix inverse) operations on the set of coefficients. The computational growth in space dimension is only n^3, as opposed to D^n for grid-based methods, where D is the number of grid-points per space dimension, and the solution is only obtained over a finite region in the latter case. The time-step is τ, and the total time propagation at step N is $T = N\tau$. The errors go to zero as $\tau \downarrow 0$ and $T \to \infty$. The specific error bounds are as follows. For an error on the order of $\varepsilon(1+|x|^2)$ over all of $\mathbb{R}^n$, it is sufficient to have $\tau \propto \varepsilon^2$ and $N \propto \varepsilon^{-3}$.

This elimination of the curse-of-dimensionality alone was significant, but there was a terrible price to be paid in the above basic algorithm in terms of what we call the curse-of-complexity. In particular, $K_N \doteq \#\mathcal{K}_N$ grows like M^N. However, it was quickly noticed that, in practice, the great majority of the quadratics, U_k^N, typically contributed little or nothing to the actual solution. Various pruning strategies were applied to attenuate this complexity growth through the elimination of less-valuable quadratics. (Essentially, these project the solution onto a lower-dimensional abstract space at each step.) However, during a visit with Dr. S. Gaubert at INRIA, we determined that certain tools from convex optimization could be applied to radically improve this pruning process. See [20]. With this tool in hand we demonstrated solution of an HJB PDE over all of $\mathbb{R}^6$ with $M = 6$ in roughly 45 minutes on a desktop machine. This would require many years of CPU time with standard methods (solving only over a relatively large, but finite, region in the latter case). Of course, one cannot depict a solution over 6-dimensional space. However, some data along planes is depicted in Figures 1–2.

The above results require that the Hamiltonian take the form of a maximum of quadratic forms. In order to make the approach amenable to general nonlinear problems, one must determine a means for approximating a general (semiconvex) Hamiltonian as a maximum of quadratic forms. In order to

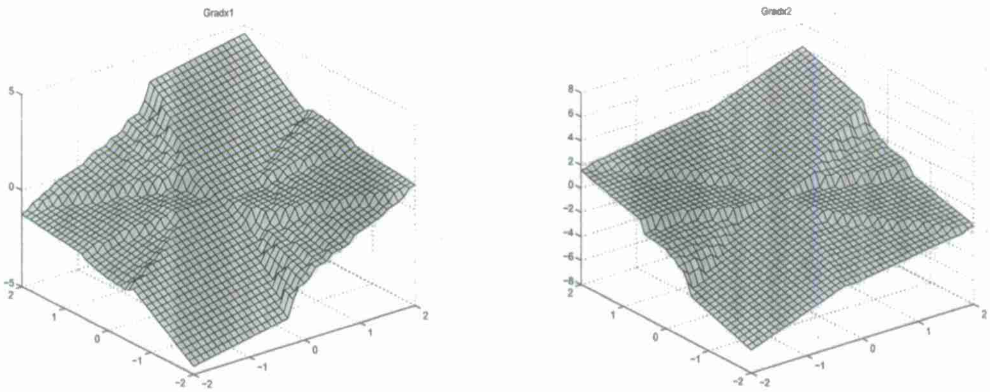

Figure 1: x_1 and x_2 partials on the x_1, x_2 plane

do this sensibly, one needs to determine a mapping from the approximation quality of the Hamiltonian to the resulting error in the solution. This will lead us to appropriate approximations of a general Hamiltonian. In [17], [18], we demonstrate such a mapping.

Value-Based Sensor Tasking

The correct measure for sensor UAV tasking is the expected payoff to the warfighters of the possible observational data returns. More typical approaches include entropy-based metrics, but these are not *correct* in the sense that they do not represent the actual value of the information to the warfighter. For example, one location might be a far more dangerous potential location for opposing forces to be firing from than another. In [23], McEneaney developed an object, $V_\tau(q)$ which describes the minimax, expected payoff for a game between Blue and Red at time t as a function of the Blue knowledge of the system state, specified as probability distribution q. (It is assumed that Red has perfect state information.)

This object may be used to determine the value of sensing actions. For

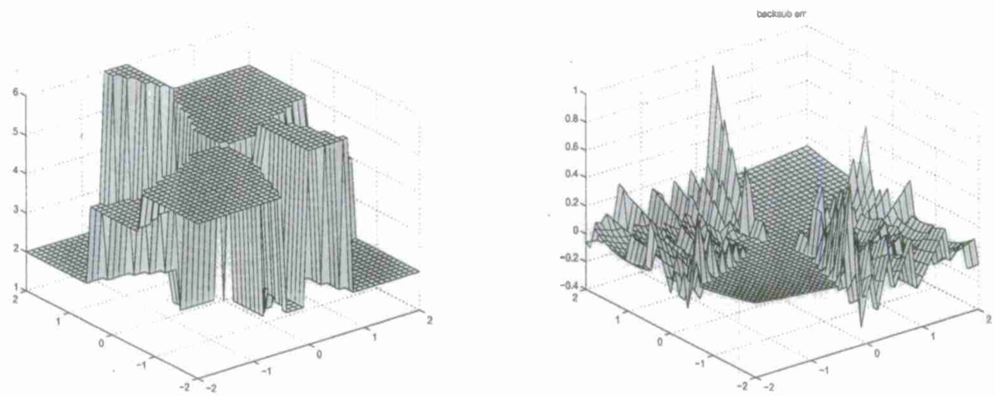

Figure 2: Optimal switching and backsubstitution error on the x_1, x_2 plane

discussion purposes, suppose a simple, decomposed problem is given as follows. Suppose Blue will choose a sensing control action, ending at time, $t = \tau > 0$, immediately followed by a combat action. At time, $t = 0$, Blue knowledge is described by q_0. Given a series of sensing actions, $\{u_t\}_{t=0}^{\tau}$, there is an associated set of possible observations, $\{y_t(u_t)\}_{t=0}^{\tau}$. Note that the y_t are random variables - the actual observation that will be obtained will be corrupted by noise. Given such a set of observations, one may update the distribution q_0 to q_τ by an estimator such as Bayes rule. Note that q_τ is a random variable. Let the resulting expected payoff assuming optimal future troop actions be denoted by $V(\tau, q_\tau)$.

Based on an analysis of the combat actions, in the case where the opposing force stays fixed in their possibly hidden urban locations, we were able to show that this value will always take the form of a pointwise maximum of linear functionals over the probability simplex. That is, one has

$$V(\tau, q) = \max_{i \in \mathcal{I}_\tau} b_\tau^i \cdot q.$$

One also has the usual dynamic programming result that for $t \in \{0, 1, \dots \tau -$

1},

$$V(t,q) = \max_{u \in \mathcal{U}} \mathbf{E}_y \left\{ V(t+1, \beta^{y,u}(q)) \right\}$$

where the expectation is over the set of possible observations.

Very interestingly, we found that if $V(t+1,q)$ takes the form $V(t+1,q) = \max_{i \in \mathcal{I}_{t+1}} b^i_{t+1} \cdot q$ where $\mathcal{I}_{t+1} = \{1, 2, \dots I_{t+1}\}$. Then,

$$V(t,q) = \max_{i \in \mathcal{I}_t} b^i_t \cdot q$$

where $\mathcal{I}_t = \{1, 2, \dots I_t\}$, $I_t = N_u (I_{t+1})^{N_y}$,

$$b^i_t = \sum_{y \in \mathcal{Y}} D(\mathbf{R}^{y,u}) b^{j_y}_{t+1} \tag{1}$$

where $(u, \{j_y\}) = \mathcal{M}^{-1}(i)$, and $\mathcal{M}$ is a one-to-one, onto mapping from $\mathcal{U} \times \mathcal{P}^{N_y}(\mathcal{I}_{t+1}) \to \mathcal{I}_t$ (i.e., an indexing of $\mathcal{U} \times \mathcal{P}^{N_y}(\mathcal{I}_{t+1})$). Here, $D(\mathbf{R}^{y,u})$ is a Bayes rule updater; the linearity is a result of an expectation operation. Using this technology, we began solving sensing control problems. A very simple example of an optimal path can be seen in the right-hand image of Figure 3, where the blue path indicates movement of combat ground forces, and the green path indicates the optimal tasking path of a single supporting sensor platform. More information on this area of research appears in [25], [26].

At some later time, we realized the full implication of this result. It was that one essentially had a max-plus curse-of-dimensionality-free approach to a stochastic control problem. This was entirely unexpected, as it was previously believed that max-plus linearity was required for max-plus curse-of-dimensionality-free methods. However, in fact, what is actually required is quite a bit less. We are now exploring idempotent (max-plus and min-max) curse-of-dimensionality-free methods for stochastic control problems and certain games. This is the beginning of a new branch of research in application of idempotent methods to control, estimation, games and Hamilton-Jacobi PDEs; see [11], [12], [13], [14].

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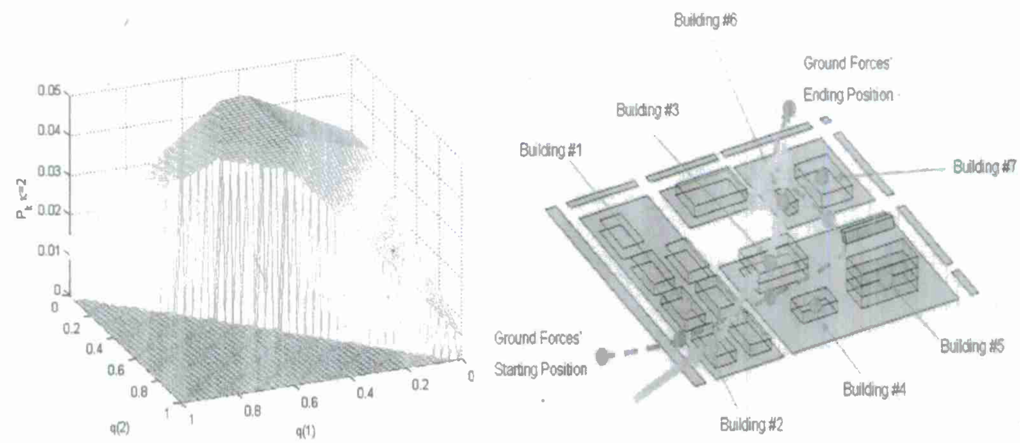

Figure 3: $-V_t(q)$ for a micro-action and optimal sensing-support route

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